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Optimal bounds for Neuman means in terms of geometric, arithmetic and quadratic means

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Abstract
In this paper, we present sharp bounds for the two Neuman means S_{HA} and S_{CA} derived from the Schwab-Borchardt mean in terms of convex combinations of either the weighted arithmetic and geometric means or the weighted arithmetic and quadratic means, and the mean generated either by the geometric or by the quadratic mean.

MSC: 26E60

Keywords: Schwab-Borchardt mean; Neuman mean; geometric mean; arithmetic mean; quadratic mean

1 Introduction

Let $a, b > 0$ with $a \neq b$, then the Schwab-Borchardt mean $SB(a,b)$ is defined by

$$SB(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases} \tag{1.1}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well known that $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean, for example,

$$P(a,b) = \frac{a-b}{2\sin^{-1}[(a-b)/(a+b)]} = SB(G,A) \quad \text{is the first Seiffert mean,}$$

$$T(a,b) = \frac{a-b}{2\tan^{-1}[(a-b)/(a+b)]} = SB(A,Q) \quad \text{is the second Seiffert mean,}$$

$$M(a,b) = \frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]} = SB(Q,A) \quad \text{is the Neuman-Sándor mean,}$$

$$L(a,b) = \frac{a-b}{2\tanh^{-1}[(a-b)/(a+b)]} = SB(A,G) \quad \text{is the logarithmic mean,}$$

where $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$ and $Q(a,b) = \sqrt{(a^2+b^2)/2}$ denote the classical geometric mean, arithmetic mean and quadratic mean of a and b, respectively. The Schwab-Borchardt mean $SB(a,b)$ was investigated in [1, 2].

Let $H(a,b)=2ab/(a+b)$, $C(a,b)=(a^2+b^2)/(a+b)$ be the harmonic and contraharmonic means of two positive numbers a and b, respectively. Then it is well known that

$$\begin{aligned} &H(a,b) < G(a,b) < L(a,b) < P(a,b) \\ &\quad < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b) \end{aligned} \tag{1.2}$$

for $a,b>0$ with $a \neq b$.

Recently, the Schwab-Borchardt mean and its special cases have been the subject of intensive research. Neuman and Sándor [3, 4] proved that the inequalities

$$\begin{aligned}
&P(a,b) > \frac{2}{\pi}A(a,b), \qquad \frac{A(a,b)}{\log(1+\sqrt{2})} > M(a,b) > \frac{\pi}{4\log(1+\sqrt{2})}T(a,b), \\
&T\bigl(A(a,b),G(a,b)\bigr) < P(a,b), \qquad T(a,b) > T\bigl(A(a,b),Q(a,b)\bigr), \\
&L(a,b) < L\bigl(A(a,b),G(a,b)\bigr), \qquad M(a,b) < L\bigl(A(a,b),Q(a,b)\bigr), \\
&L(a,b) > H\bigl(P(a,b),G(a,b)\bigr), \qquad P(a,b) > H\bigl(L(a,b),A(a,b)\bigr), \\
&M(a,b) > H\bigl(T(a,b),A(a,b)\bigr), \qquad T(a,b) > H\bigl(M(a,b),Q(a,b)\bigr), \\
&G(a,b)P(a,b) < L^2(a,b) < \frac{G^2(a,b)+P^2(a,b)}{2}, \\
&L(a,b)A(a,b) < P^2(a,b) < \frac{L^2(a,b)+A^2(a,b)}{2}, \\
&A(a,b)T(a,b) < M^2(a,b) < \frac{A^2(a,b)+T^2(a,b)}{2}, \\
&M(a,b)Q(a,b) < T^2(a,b) < \frac{M^2(a,b)+Q^2(a,b)}{2}, \\
&Q^{1/3}(a,b)A^{2/3}(a,b) < M(a,b) < \frac{1}{3}Q(a,b) + \frac{2}{3}A(a,b)
\end{aligned}$$

hold for all $a,b>0$ with $a \neq b$. In [5], the author proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b)$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\ldots$, $\beta \geq 1/3$, $\lambda \leq [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\ldots$ and $\mu \geq 1/6$. Chu and Long [6] found that the double inequality

$$M_p(a,b) < M(a,b) < qI(a,b)$$

holds for all $a,b>0$ with $a \neq b$ if and only if $p \leq \log 2/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ and $q \geq e/[2\log(1+\sqrt{2})] = 1.5420\ldots$, where $M_p(a,b) = [(a^p+b^p)/2]^{1/p}$ $(p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ is the pth power mean of a and b. Zhao *et al.* [7] presented the least values α_1, α_2, α_3 and the greatest values β_1, β_2, β_3 such that the double inequalities

$$\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b) + (1-\beta_1)Q(a,b),$$
$$\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b) + (1-\beta_2)Q(a,b)$$

and

$$\alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b) + (1-\beta_3)C(a,b)$$

hold for all $a,b > 0$ with $a \neq b$.

Very recently, the bivariate means S_{AH}, S_{HA}, S_{CA} and S_{AC} derived from the Schwab-Borchardt mean have been defined by Neuman [8, 9] as follows:

$$S_{AH} = SB(A,H), \qquad S_{HA} = SB(H,A), \qquad S_{CA} = SB(C,A), \qquad S_{AC} = SB(A,C). \tag{1.3}$$

We call the means S_{AH}, S_{HA}, S_{CA} and S_{AC} given in (1.3) the Neuman means. Moreover, let $v = (a-b)/(a+b) \in (-1,1)$, then the following explicit formulas for S_{AH}, S_{HA}, S_{AC} and S_{CA} have been found by Neuman [8]:

$$S_{AH} = A\frac{\tanh(p)}{p}, \qquad S_{HA} = A\frac{\sin(q)}{q}, \tag{1.4}$$
$$S_{CA} = A\frac{\sinh(r)}{r}, \qquad S_{AC} = A\frac{\tan(s)}{s}, \tag{1.5}$$

where p, q, r and s are defined implicitly as $\operatorname{sech}(p) = 1 - v^2$, $\cos(q) = 1 - v^2$, $\cosh(r) = 1 + v^2$ and $\sec(s) = 1 + v^2$, respectively. Clearly, $p \in (0,\infty)$, $q \in (0,\pi/2)$, $r \in (0, \log(2+\sqrt{3}))$ and $s \in (0,\pi/3)$.

In [8], Neuman proved that the inequalities

$$H(a,b) < S_{AH}(a,b) < L(a,b) < S_{HA}(a,b) < P(a,b), \tag{1.6}$$
$$T(a,b) < S_{CA}(a,b) < Q(a,b) < S_{AC}(a,b) < C(a,b) \tag{1.7}$$

hold for $a,b > 0$ with $a \neq b$.

He *et al.* [10] found the greatest values $\alpha_1, \alpha_2 \in [0,1/2]$, $\alpha_3, \alpha_4 \in [1/2,1]$ and the least values $\beta_1, \beta_2 \in [0,1/2]$, $\beta_3, \beta_4 \in [1/2,1]$ such that the double inequalities

$$H\big(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a\big) < S_{AH}(a,b) < H\big(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a\big),$$
$$H\big(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a\big) < S_{HA}(a,b) < H\big(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a\big),$$
$$C\big(\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a\big) < S_{CA}(a,b) < C\big(\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a\big)$$

and

$$C\big(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a\big) < S_{AC}(a,b) < C\big(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a\big)$$

hold for all $a,b > 0$ with $a \neq b$.

It follows from (1.2) and (1.6) together with (1.7) that

$$G(a,b) < S_{HA}(a,b) < A(a,b) < S_{CA}(a,b) < Q(a,b) \tag{1.8}$$

for all $a, b > 0$ with $a \neq b$.

For fixed $a, b > 0$ with $a \neq b$, let $x \in [0, 1/2]$, $y \in [1/2, 1]$,

$$f(x) = G\bigl[xa + (1-x)b, xb + (1-x)a\bigr], \tag{1.9}$$

$$g(y) = Q\bigl[ya + (1-y)b, yb + (1-y)a\bigr]. \tag{1.10}$$

Then it is not difficult to verify that $f(x)$ and $g(y)$ are continuous and strictly increasing on $[0, 1/2]$ and $[1/2, 1]$, respectively. Note that

$$f(0) = G(a,b) < S_{HA}(a,b) < A(a,b) = f(1/2), \tag{1.11}$$

$$g(1/2) = A(a,b) < S_{CA}(a,b) < Q(a,b) = g(1). \tag{1.12}$$

Motivated by (1.8)-(1.12), in the article we present the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, $\alpha_3, \beta_3 \in [0, 1/2]$ and $\alpha_4, \beta_4 \in [1/2, 1]$ such that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)G(a,b) < S_{HA}(a,b) < \beta_1 A(a,b) + (1-\beta_1)G(a,b),$$

$$\alpha_2 A(a,b) + (1-\alpha_2)Q(a,b) < S_{CA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)Q(a,b),$$

$$G\bigl[\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a\bigr] < S_{HA}(a,b) < G\bigl[\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a\bigr],$$

$$Q\bigl[\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a\bigr] < S_{CA}(a,b) < Q\bigl[\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a\bigr]$$

hold for all $a, b > 0$ with $a \neq b$.

Our main results are the following Theorems 1.1-1.4. All numerical computations are carried out using Mathematica software.

Theorem 1.1 *The double inequality*

$$\alpha_1 A(a,b) + (1-\alpha_1)G(a,b) < S_{HA}(a,b) < \beta_1 A(a,b) + (1-\beta_1)G(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$ and $\beta_1 \geq 2/\pi$.

Theorem 1.2 *The two-sided inequality*

$$\alpha_2 A(a,b) + (1-\alpha_2)Q(a,b) < S_{CA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)Q(a,b)$$

holds true for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \geq 1/3$ and $\beta_2 \leq [\sqrt{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(\sqrt{2}-1)\log(2+\sqrt{3})] = 0.2390\ldots$.

Theorem 1.3 *Let $\alpha_3, \beta_3 \in [0, 1/2]$, then the double inequality*

$$G\bigl[\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a\bigr] < S_{HA}(a,b) < G\bigl[\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a\bigr]$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_3 \leq 1/2 - \sqrt{6}/6 = 0.09175\ldots$ *and* $\beta_3 \geq 1/2 - \sqrt{\pi^2 - 4}/(2\pi) = 0.1144\ldots$.

Theorem 1.4 *Let* $\alpha_4, \beta_4 \in [1/2, 1]$, *then the two-sided inequality*

$$Q\big[\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a\big] < S_{CA}(a,b) < Q\big[\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a\big]$$

holds true for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_4 \leq 1/2 + \sqrt{6}/6 = 0.9082\ldots$ *and* $\beta_4 \geq 1/2 + \sqrt{3/[\log(2+\sqrt{3})]^2 - 1/2} = 0.9271\ldots$.

2 Two lemmas

In order to prove our main results, we need two lemmas, which we present in this section.

Lemma 2.1 *Let* $p \in \mathbb{R}$ *and*

$$f(x) = (1-p)x^3 + \big(-2p^2 + 5p - 1\big)x^2 + \big(2p^2 + p - 1\big)x + p - 1. \tag{2.1}$$

Then the following statements are true:

(1) *If* $p = 1/3$, *then* $f(x) < 0$ *for all* $x \in (0,1)$ *and* $f(x) > 0$ *for all* $x \in (1, \sqrt{2})$;
(2) *If* $p = 2/\pi$, *then there exists* $\lambda_1 \in (0,1)$ *such that* $f(x) < 0$ *for* $x \in (0, \lambda_1)$ *and* $f(x) > 0$ *for* $x \in (\lambda_1, 1)$;
(3) *If* $p = [\sqrt{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(\sqrt{2}-1)\log(2+\sqrt{3})]$, *then there exists* $\lambda_2 \in (1, \sqrt{2})$ *such that* $f(x) < 0$ *for* $x \in (1, \lambda_2)$ *and* $f(x) > 0$ *for* $x \in (\lambda_2, \sqrt{2})$.

Proof For part (1), if $p = 1/3$, then (2.1) becomes

$$f(x) = \frac{2}{9}(x-1)\big(3x^2 + 5x + 3\big). \tag{2.2}$$

Therefore, part (1) follows easily from (2.2).

For part (2), if $p = 2/\pi$, then simple computations lead to

$$-2p^2 + 5p - 1 = \frac{-\pi^2 + 10\pi - 8}{\pi^2} > 0, \tag{2.3}$$

$$2p^2 + p - 1 = \frac{-\pi^2 + 2\pi + 8}{\pi^2} > 0, \tag{2.4}$$

$$f(0) = -\frac{\pi - 2}{\pi} < 0, \tag{2.5}$$

$$f(1) = \frac{2(6-\pi)}{\pi} > 0, \tag{2.6}$$

$$f'(x) = 3(1-p)x^2 + 2\big(-2p^2 + 5p - 1\big)x + \big(2p^2 + p - 1\big). \tag{2.7}$$

It follows from (2.3) and (2.4) together with (2.7) that $f(x)$ is strictly increasing on $(0,1)$. Therefore, part (2) follows from (2.5) and (2.6) together with the monotonicity of $f(x)$.

For part (3), if $p = [\sqrt{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(\sqrt{2}-1)\log(2+\sqrt{3})] = 0.2390\ldots$, then numerical computations lead to

$$-2p^2 + 5p - 1 = 0.0810\ldots > 0, \tag{2.8}$$

$$f(1) = -0.5656\ldots < 0, \tag{2.9}$$

$$f(\sqrt{2}) = 0.6388\ldots > 0. \tag{2.10}$$

It follows from (2.7) and (2.8) that

$$f'(x) > 3(1-p) + 2\left(-2p^2 + 5p - 1\right) + \left(2p^2 + p - 1\right) = 2p(4-p) > 0 \tag{2.11}$$

for $x \in (1, \sqrt{2})$.

Therefore, part (3) follows easily from (2.9)-(2.11). □

Lemma 2.2 *Let $p \in \mathbb{R}$ and*

$$\begin{aligned} g(x) = {} & (2p-1)^4 x^3 + \left(-256p^6 + 768p^5 - 1{,}008p^4 + 736p^3 - 296p^2 + 56p - 3\right)x^2 \\ & + \left(512p^6 - 1{,}536p^5 + 1{,}776p^4 - 992p^3 + 248p^2 - 8p - 1\right)x \\ & + \left(-256p^6 + 768p^5 - 784p^4 + 288p^3 - 24p^2 + 8p - 1\right). \end{aligned} \tag{2.12}$$

Then the following statements are true:

(1) *If $p = 1/2 - \sqrt{6}/6$, then $g(x) < 0$ for all $x \in (0,1)$;*
(2) *If $p = 1/2 + \sqrt{6}/6$, then $g(x) > 0$ for all $x \in (1,2)$;*
(3) *If $p = 1/2 - \sqrt{\pi^2 - 4}/(2\pi)$, then there exists $\lambda_3 \in (0,1)$ such that $g(x) < 0$ for $x \in (0, \lambda_3)$ and $g(x) > 0$ for $x \in (\lambda_3, 1)$;*
(4) *If $p = 1/2 + \sqrt{3/[\log(2+\sqrt{3})]^2 - 1}/2$, then there exists $\lambda_4 \in (1,2)$ such that $g(x) < 0$ for $x \in (1, \lambda_4)$ and $g(x) > 0$ for $x \in (\lambda_4, 2)$.*

Proof For parts (1) and (2), if $p = 1/2 - \sqrt{6}/6$ or $p = 1/2 + \sqrt{6}/6$, then (2.12) becomes

$$g(x) = \frac{4}{27}(x-1)\left(3x^2 + 4x + 2\right). \tag{2.13}$$

Therefore, parts (1) and (2) follow from (2.13).

For part (3), if $p = 1/2 - \sqrt{\pi^2 - 4}/(2\pi)$, then numerical computations show that

$$\begin{aligned} & -256p^6 + 768p^5 - 1{,}008p^4 + 736p^3 - 296p^2 + 56p - 3 \\ & \quad = \frac{-3\pi^6 + 56\pi^4 - 240\pi^2 + 256}{\pi^6} > 0, \end{aligned} \tag{2.14}$$

$$\begin{aligned} & 512p^6 - 1{,}536p^5 + 1{,}776p^4 - 992p^3 + 248p^2 - 8p - 1 \\ & \quad = \frac{-\pi^6 - 8\pi^4 + 240\pi^2 - 512}{\pi^6} > 0, \end{aligned} \tag{2.15}$$

$$g(0) = \frac{-\pi^6 + 8\pi^4 - 16\pi^2 + 256}{\pi^6} < 0, \tag{2.16}$$

$$g(1) = \frac{4(12 - \pi^2)}{\pi^2} > 0, \tag{2.17}$$

$$\begin{aligned} g'(x) = {} & 3(2p-1)^4 x^2 + 2\left(-256p^6 + 768p^5 - 1{,}008p^4 + 736p^3 - 296p^2 + 56p - 3\right)x \\ & + \left(512p^6 - 1{,}536p^5 + 1{,}776p^4 - 992p^3 + 248p^2 - 8p - 1\right). \end{aligned} \tag{2.18}$$

From (2.14), (2.15) and (2.18) we clearly see that $g(x)$ is strictly increasing on $(0,1)$. Therefore, part (3) follows from (2.16) and (2.17) together with the monotonicity of $g(x)$.

For part (4), if $p = 1/2 + \sqrt{3/[\log(2+\sqrt{3})]^2 - 1/2}$, then numerical computations lead to

$$-256p^6 + 768p^5 - 1{,}008p^4 + 736p^3 - 296p^2 + 56p - 3 = -0.2329\ldots < 0, \tag{2.19}$$

$$512p^6 - 1{,}536p^5 + 1{,}776p^4 - 992p^3 + 248p^2 - 8p - 1 = -0.6027\ldots < 0, \tag{2.20}$$

$$g(1) = -0.7567\ldots < 0, \tag{2.21}$$

$$g(2) = 1.6692\ldots > 0, \tag{2.22}$$

$$-48p^4 + 96p^3 - 68p^2 + 20p - 1 = 0.1322\ldots > 0. \tag{2.23}$$

It follows from (2.18), (2.19), (2.20) and (2.23) that

$$\begin{aligned} g'(x) &> 3(2p-1)^4x^2 + 2\big(-256p^6 + 768p^5 - 1{,}008p^4 + 736p^3 - 296p^2 + 56p - 3\big)x^2 \\ &\quad + \big(512p^6 - 1{,}536p^5 + 1{,}776p^4 - 992p^3 + 248p^2 - 8p - 1\big)x^2 \\ &= 4\big(-48p^4 + 96p^3 - 68p^2 + 20p - 1\big)x^2 > 0 \end{aligned} \tag{2.24}$$

for $x \in (1,2)$.

Therefore, part (4) follows from (2.21) and (2.22) together with (2.24). □

3 Proofs of Theorems 1.1-1.4

Proof of Theorem 1.1 Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\lambda = v\sqrt{2-v^2}$, $x = \sqrt[4]{1-\lambda^2}$ and $p \in \{1/3, 2/\pi\}$. Then $v, \lambda, x \in (0,1)$ and (1.4) leads to

$$\frac{S_{HA}(a,b) - G(a,b)}{A(a,b) - G(a,b)} = \frac{\lambda - (1-\lambda^2)^{1/4}\sin^{-1}(\lambda)}{[1-(1-\lambda^2)^{1/4}]\sin^{-1}(\lambda)}, \tag{3.1}$$

$$\begin{aligned} &S_{HA}(a,b) - \big[pA(a,b) + (1-p)G(a,b)\big] \\ &\quad = A(a,b)\left[\frac{\lambda}{\sin^{-1}(\lambda)} - (1-p)\big(1-\lambda^2\big)^{1/4} - p\right] \\ &\quad = \frac{A(a,b)[p + (1-p)(1-\lambda^2)^{1/4}]}{\sin^{-1}(\lambda)}F(x), \end{aligned} \tag{3.2}$$

where

$$F(x) = \frac{\sqrt{1-x^4}}{(1-p)x + p} - \sin^{-1}\big(\sqrt{1-x^4}\big),$$

$$F(0) = \frac{1}{p} - \frac{\pi}{2}, \tag{3.3}$$

$$F(1) = 0 \tag{3.4}$$

and

$$F'(x) = \frac{1-x}{\sqrt{1-x^4}[(1-p)x+p]^2}f(x), \tag{3.5}$$

where $f(x)$ is defined as in Lemma 2.1.

We divide the proof into two cases.

Case 1: $p = 1/3$. Then from Lemma 2.1(1) and (3.5) we clearly see that $F(x)$ is strictly decreasing on $(0,1)$. Therefore,

$$S_{HA}(a,b) > \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) \tag{3.6}$$

for all $a,b > 0$ with $a \neq b$ follows from (3.2) and (3.4) together with the monotonicity of $F(x)$.

Case 2: $p = 2/\pi$. Then from (3.3), (3.5) and Lemma 2.1(2) we know that

$$F(0) = 0 \tag{3.7}$$

and there exists $\lambda_1 \in (0,1)$ such that $F(x)$ is strictly decreasing on $(0,\lambda_1]$ and strictly increasing on $[\lambda_1,1)$. Therefore,

$$S_{HA}(a,b) < \frac{2}{\pi}A(a,b) + \left(1 - \frac{2}{\pi}\right)G(a,b) \tag{3.8}$$

for all $a,b > 0$ with $a \neq b$ follows from (3.2) and (3.4) together with (3.7) and the piecewise monotonicity of $F(x)$.

Note that

$$\lim_{\lambda\to 0^+} \frac{\lambda - (1-\lambda^2)^{1/4}\sin^{-1}(\lambda)}{[1-(1-\lambda^2)^{1/4}]\sin^{-1}(\lambda)} = \frac{1}{3} \tag{3.9}$$

and

$$\lim_{\lambda\to 1^-} \frac{\lambda - (1-\lambda^2)^{1/4}\sin^{-1}(\lambda)}{[1-(1-\lambda^2)^{1/4}]\sin^{-1}(\lambda)} = \frac{2}{\pi}. \tag{3.10}$$

Therefore, Theorem 1.1 follows from (3.6) and (3.8) together with the following statements.

- If $\alpha > 1/3$, then equations (3.1) and (3.9) imply that there exists small enough $\delta > 0$ such that $S_{HA}(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b)$ for all $a > b > 0$ with $b/a \in (1-\delta,1)$.
- If $\beta < 2/\pi$, then equations (3.1) and (3.10) imply that there exists large enough $M > 1$ such that $S_{HA}(a,b) > \beta A(a,b) + (1-\beta)G(a,b)$ for all $a > b > 0$ with $a/b \in (M,+\infty)$. □

Proof of Theorem 1.2 Without loss of generality, we assume that $a > b$. Let $\nu = (a-b)/(a+b)$, $\mu = \nu\sqrt{2+\nu^2}$, $x = \sqrt[4]{1+\mu^2}$ and $p \in \{[\sqrt{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(\sqrt{2}-1)\log(2+\sqrt{3})], 1/3\}$. Then $\nu \in (0,1)$, $\mu \in (0,\sqrt{3})$, $x \in (1,\sqrt{2})$ and (1.5) leads to

$$\frac{S_{CA}(a,b) - Q(a,b)}{A(a,b) - Q(a,b)} = \frac{\mu - (1+\mu^2)^{1/4}\sinh^{-1}(\mu)}{[1-(1+\mu^2)^{1/4}]\sinh^{-1}(\mu)}, \tag{3.11}$$

$$\begin{aligned} &S_{CA}(a,b) - \left[pA(a,b) + (1-p)Q(a,b)\right] \\ &\quad = A(a,b)\left[\frac{\mu}{\sinh^{-1}(\mu)} - (1-p)\left(1+\mu^2\right)^{1/4} - p\right] \\ &\quad = \frac{A(a,b)[(1-p)(1+\mu^2)^{1/4} + p]}{\sinh^{-1}(\mu)}G(x), \end{aligned} \tag{3.12}$$

where

$$G(x) = \frac{\sqrt{x^4-1}}{(1-p)x+p} - \sinh^{-1}\bigl(\sqrt{x^4-1}\bigr),$$

$$G(1) = 0, \tag{3.13}$$

$$G(\sqrt{2}) = \frac{\sqrt{3}}{\sqrt{2}-(\sqrt{2}-1)p} - \log(2+\sqrt{3}), \tag{3.14}$$

$$G'(x) = \frac{x-1}{\sqrt{x^4-1}[(1-p)x+p]^2} f(x), \tag{3.15}$$

where $f(x)$ is defined as in Lemma 2.1.

We divide the proof into two cases.

Case 1: $p = [\sqrt{2}\log(2+\sqrt{3})-\sqrt{3}]/[(\sqrt{2}-1)\log(2+\sqrt{3})] = 0.2390\ldots$. Then from (3.14) and (3.15) together with Lemma 2.1(3) we clearly see that there exists $\lambda_2 \in (1,\sqrt{2})$ such that $G(x)$ is strictly decreasing on $(1,\lambda_2]$ and strictly increasing on $[\lambda_2,\sqrt{2})$, and

$$G(\sqrt{2}) = 0. \tag{3.16}$$

Therefore,

$$\begin{aligned} S_{CA}(a,b) < {} & \frac{\sqrt{2}\log(2+\sqrt{3})-\sqrt{3}}{(\sqrt{2}-1)\log(2+\sqrt{3})} A(a,b) \\ & + \frac{\sqrt{3}-\log(2+\sqrt{3})}{(\sqrt{2}-1)\log(2+\sqrt{3})} Q(a,b) \end{aligned} \tag{3.17}$$

for all $a,b>0$ with $a \neq b$ follows easily from (3.12) and (3.13) together with (3.16) and the piecewise monotonicity of $G(x)$.

Case 2: $p = 1/3$. Then Lemma 2.1(1) and (3.15) lead to the conclusion that $G(x)$ is strictly increasing on $(1,\sqrt{2})$. Therefore,

$$S_{CA}(a,b) > \frac{1}{3}A(a,b) + \frac{2}{3}Q(a,b) \tag{3.18}$$

for all $a,b>0$ with $a \neq b$ follows from (3.12) and (3.13) together with the monotonicity of $G(x)$.

Note that

$$\lim_{\mu\to 0^+} \frac{\mu-(1+\mu^2)^{1/4}\sinh^{-1}(\mu)}{[1-(1+\mu^2)^{1/4}]\sinh^{-1}(\mu)} = \frac{1}{3} \tag{3.19}$$

and

$$\lim_{\mu\to\sqrt{3}^-} \frac{\mu-(1+\mu^2)^{1/4}\sinh^{-1}(\mu)}{[1-(1+\mu^2)^{1/4}]\sinh^{-1}(\mu)} = \frac{\sqrt{2}\log(2+\sqrt{3})-\sqrt{3}}{(\sqrt{2}-1)\log(2+\sqrt{3})}. \tag{3.20}$$

Therefore, Theorem 1.2 follows from (3.11) and (3.17)-(3.20). □

Proof of Theorem 1.3 Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\lambda = v\sqrt{2-v^2}$, $x = \sqrt{1-\lambda^2}$ and $p \in [0, 1/2]$. Then $v, \lambda, x \in (0, 1)$ and (1.4) leads to

$$\begin{aligned} &G\bigl[pa+(1-p)b, pb+(1-p)a\bigr] - S_{HA}(a,b) \\ &\quad = A(a,b)\left[\sqrt{1-(1-2p)^2\bigl(1-\sqrt{1-\lambda^2}\bigr)} - \frac{\lambda}{\sin^{-1}(\lambda)}\right] \\ &\quad = \frac{A(a,b)\sqrt{1-(1-2p)^2(1-\sqrt{1-\lambda^2})}}{\sin^{-1}(\lambda)} H(x), \end{aligned} \tag{3.21}$$

where

$$H(x) = \sin^{-1}\bigl(\sqrt{1-x^2}\bigr) - \frac{\sqrt{1-x^2}}{\sqrt{(1-2p)^2x-(1-2p)^2+1}},$$

$$H(1) = 0, \tag{3.22}$$

$$H(0) = \frac{\pi}{2} - \frac{1}{\sqrt{1-(1-2p)^2}} \tag{3.23}$$

and

$$H'(x) = \frac{h(x)}{2\sqrt{1-x^2}[(1-2p)^2x-(1-2p)^2+1]^{3/2}}, \tag{3.24}$$

where

$$\begin{aligned} &h(x) \\ &\quad = (1-2p)^2x^2 + 2\bigl[1-(1-2p)^2\bigr]x + (1-2p)^2 - 2\bigl[(1-2p)^2x-(1-2p)^2+1\bigr]^{3/2} \\ &\quad = \frac{(x-1)g(x)}{(1-2p)^2x^2+2[1-(1-2p)^2]x+(1-2p)^2+2[(1-2p)^2x-(1-2p)^2+1]^{3/2}}, \end{aligned} \tag{3.25}$$

where $g(x)$ is defined as in Lemma 2.2.

We divide the proof into four cases.

Case 1: $p = 1/2 - \sqrt{6}/6$. Then Lemma 2.2(1) and (3.24) together with (3.25) lead to the conclusion that $H(x)$ is strictly increasing on $(0, 1)$. Therefore,

$$S_{HA}(a,b) > G\left[\left(\frac{1}{2}-\frac{\sqrt{6}}{6}\right)a + \left(\frac{1}{2}+\frac{\sqrt{6}}{6}\right)b, \left(\frac{1}{2}-\frac{\sqrt{6}}{6}\right)b + \left(\frac{1}{2}+\frac{\sqrt{6}}{6}\right)a\right]$$

for all $a, b > 0$ with $a \neq b$ follows easily from (3.21) and (3.22) together with the monotonicity of $H(x)$.

Case 2: $1/2 - \sqrt{6}/6 < p \le 1/2$. Let $q = (1-2p)^2$ and $\lambda \to 0^+$, then $0 \le q < 2/3$ and power series expansions lead to

$$\begin{aligned} &\sqrt{1-(1-2p)^2\bigl(1-\sqrt{1-\lambda^2}\bigr)} - \frac{\lambda}{\sin^{-1}\lambda} \\ &\quad = \frac{\sqrt{1-q(1-\sqrt{1-\lambda^2})}\sin^{-1}\lambda - \lambda}{\sin^{-1}\lambda} = \frac{1}{\sin^{-1}\lambda}\left[\left(\frac{1}{6}-\frac{q}{4}\right)\lambda^3 + o\bigl(\lambda^3\bigr)\right]. \end{aligned} \tag{3.26}$$

Equations (3.21) and (3.26) imply that there exists small enough $\delta_1 > 0$ such that $S_{HA}(a,b) < G[pa+(1-p)b, pb+(1-p)a]$ for all $a, b > 0$ with $b/a \in (1-\delta_1, 1)$.

Case 3: $p = 1/2 - \sqrt{\pi^2 - 4}/(2\pi)$. Then from Lemma 2.2(3) and (3.23)-(3.25) we clearly see that there exists $\lambda_3 \in (0,1)$ such that $H(x)$ is strictly increasing on $(0, \lambda_3]$ and strictly decreasing on $[\lambda_3, 1)$, and

$$H(0) = 0. \tag{3.27}$$

Therefore,

$$S_{HA}(a,b) < G\left[\left(\frac{1}{2} - \frac{\sqrt{\pi^2 - 4}}{2\pi}\right)a + \left(\frac{1}{2} + \frac{\sqrt{\pi^2 - 4}}{2\pi}\right)b, \left(\frac{1}{2} - \frac{\sqrt{\pi^2 - 4}}{2\pi}\right)b + \left(\frac{1}{2} + \frac{\sqrt{\pi^2 - 4}}{2\pi}\right)a\right]$$

for all $a, b > 0$ with $a \neq b$ follows easily from (3.21) and (3.22) together with (3.27) and the piecewise monotonicity of $H(x)$.

Case 4: $0 \leq p < 1/2 - \sqrt{\pi^2 - 4}/(2\pi)$. Then

$$\lim_{\lambda \to 1^-}\left[\sqrt{1 - (1-2p)^2\left(1 - \sqrt{1 - \lambda^2}\right)} - \frac{\lambda}{\sin^{-1}(\lambda)}\right] = \sqrt{1 - (1-2p)^2} - \frac{2}{\pi} < 0. \tag{3.28}$$

Equation (3.21) and inequality (3.28) imply that there exists large enough $M_1 > 1$ such that $S_{HA}(a,b) > G[pa + (1-p)b, pb + (1-p)a]$ for all $a, b > 0$ with $a/b \in (M_1, +\infty)$. □

Proof of Theorem 1.4 Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\mu = v\sqrt{2 + v^2}$, $x = \sqrt{1 + \mu^2}$ and $p \in [1/2, 1]$. Then $v \in (0,1)$, $\mu \in (0, \sqrt{3})$, $x \in (1,2)$ and (1.5) leads to

$$\begin{aligned} &Q\bigl[pa + (1-p)b, pb + (1-p)a\bigr] - S_{CA}(a,b) \\ &\quad = A(a,b)\left[\sqrt{1 + (1-2p)^2\left(\sqrt{1+\mu^2} - 1\right)} - \frac{\mu}{\sinh^{-1}(\mu)}\right] \\ &\quad = \frac{A(a,b)\sqrt{1 + (1-2p)^2(\sqrt{1+\mu^2} - 1)}}{\sinh^{-1}(\mu)} J(x), \end{aligned} \tag{3.29}$$

where

$$J(x) = \sinh^{-1}\left(\sqrt{x^2 - 1}\right) - \frac{\sqrt{x^2 - 1}}{\sqrt{(1-2p)^2 x - (1-2p)^2 + 1}},$$

$$J(1) = 0, \tag{3.30}$$

$$J(2) = \log(2 + \sqrt{3}) - \frac{\sqrt{3}}{\sqrt{1 + (1-2p)^2}}, \tag{3.31}$$

$$\begin{aligned} J'(x) &= \frac{2[(1-2p)^2 x - (1-2p)^2 + 1]^{3/2} - [(1-2p)^2 x^2 + 2(1 - (1-2p)^2)x + (1-2p)^2]}{2\sqrt{x^2 - 1}[(1-2p)^2 x - (1-2p)^2 + 1]^{3/2}} \\ &= -\frac{1}{2[(1-2p)^2 x - (1-2p)^2 + 1]^{3/2} + [(1-2p)^2 x^2 + 2(1 - (1-2p)^2)x + (1-2p)^2]} \\ &\quad \times \frac{x - 1}{2\sqrt{x^2 - 1}[(1-2p)^2 x - (1-2p)^2 + 1]^{3/2}} g(x), \end{aligned} \tag{3.32}$$

where $g(x)$ is defined as in Lemma 2.2.

We divide the proof into four cases.

Case 1: $p = 1/2 + \sqrt{6}/6$. Then Lemma 2.2(2) and (3.32) lead to the conclusion that $J(x)$ is strictly increasing on $(1, 2)$. Therefore,

$$S_{CA}(a,b) > Q\left[\left(\frac{1}{2}+\frac{\sqrt{6}}{6}\right)a+\left(\frac{1}{2}-\frac{\sqrt{6}}{6}\right)b,\left(\frac{1}{2}+\frac{\sqrt{6}}{6}\right)b+\left(\frac{1}{2}-\frac{\sqrt{6}}{6}\right)a\right]$$

for all $a, b > 0$ with $a \neq b$ follows easily from (3.29) and (3.30) together with the monotonicity of $J(x)$.

Case 2: $1/2 + \sqrt{6}/6 < p \leq 1$. Let $q = (1-2p)^2$ and $\mu \to 0^+$, then $1 \geq q > 2/3$ and power series expansions lead to

$$\begin{aligned}
&\sqrt{1+(1-2p)^2\left(\sqrt{1+\mu^2}-1\right)}-\frac{\mu}{\sinh^{-1}(\mu)}\\
&\quad=\frac{\sqrt{1+q(\sqrt{1+\mu^2}-1)}\sinh^{-1}(\mu)-\mu}{\sinh^{-1}(\mu)}\\
&\quad=\frac{1}{\sinh^{-1}(\mu)}\left[\left(\frac{1}{4}q-\frac{1}{6}\right)\mu^3+o\left(\mu^3\right)\right].
\end{aligned}\tag{3.33}$$

Equations (3.29) and (3.33) imply that there exists small enough $\delta_2 > 0$ such that $S_{CA}(a,b) < Q[pa+(1-p)b, pb+(1-p)a]$ for all $a, b > 0$ with $b/a \in (1-\delta_2, 1)$.

Case 3: $p = 1/2 + \sqrt{3/[\log(2+\sqrt{3})]^2-1}/2$. Then (3.31) and (3.32) together with Lemma 2.2(4) lead to the conclusion that there exists $\lambda_4 \in (1, 2)$ such that $J(x)$ is strictly increasing on $(1, \lambda_4]$ and strictly decreasing on $[\lambda_4, 2)$, and

$$J(2) = 0. \tag{3.34}$$

Therefore,

$$S_{CA}(a,b) < Q\left[pa+(1-p)b, pb+(1-p)a\right]$$

for all $a, b > 0$ with $a \neq b$ follows easily from (3.29) and (3.30) together with (3.34) and the piecewise monotonicity of $J(x)$.

Case 4: $1/2 \leq p < 1/2 + \sqrt{3/[\log(2+\sqrt{3})]^2-1}/2$. Then

$$\begin{aligned}
&\lim_{\mu\to\sqrt{3}^-}\left[\sqrt{1+(1-2p)^2\left(\sqrt{1+\mu^2}-1\right)}-\frac{\mu}{\sinh^{-1}(\mu)}\right]\\
&\quad=\sqrt{1+(2p-1)^2}-\frac{\sqrt{3}}{\log(2+\sqrt{3})}<0.
\end{aligned}\tag{3.35}$$

Equation (3.29) and inequality (3.35) imply that there exists large enough $M_2 > 1$ such that $S_{CA}(a,b) > Q[pa+(1-p)b, pb+(1-p)a]$ for all $a, b > 0$ with $a/b \in (M_2, +\infty)$. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

W-MQ provided the main idea and carried out the proof of Lemmas 2.1 and 2.2. Y-MC carried out the proof of Theorems 1.1-1.4. All authors read and approved the final manuscript.

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